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Contribution of Fixed Point Theorem in Quasi Metric Spaces

Dr.Ayaz Ahmad,
Associate Professor, Department of Mathematics,
Millat College, Darbhanga, Bihar- 846004, India
ayazahmadmillatcollege@gmail.com

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Abstract: In this paper we study on contribution of fixed point theorem in Metric spaces and Quasi Metric spaces.

Key words: Metric space, Contraction Mapping, Fixed point Theorem, Quasi Metric Space, p-Convergent, p-orbit ally continuous.

Definition: 1 (**Metric Space**) Let X be a non-empty set-A function XxX →R (the set of reals) such that p:XxX→R is called a metric or distance function (if ad only if) p satisfies the following conditions.

(i) $p(x,y) \geq 0$ for all $x, y \in x$
(ii) $p(x,y)= 0$ if $x=y$
(iii) $p(x,y) = p(y,x)$ for all $x,y \in x$
(iv) $p(x,y) \geq p(x,z)+ p(z,y)$ for all $x,y,z \in X$

If p is a metric for X, then the pair (X, p) is called a metric space.

Definition: 2 (Cauchy sequence) Let (X,p) be a metric space .Then a sequence $\{x_n\}$ of points of X is said to be a Cauchy sequence if for each $\varepsilon>0$,there exists a positive integer n_0 such that $m,n\geq n_0$ implies $p(x_m,x_n) < \varepsilon$.

Definition: 3 (**Completeness**): A metric space(X,p) is said to be complete if every Cauchy sequence in X converges to point of X.

Definition: 4 (Contraction mapping) Let (X, p) be a complete metric space. A mapping T:X→X is said to be a contraction mapping if there exists a real number α with $0 <\alpha< 1$ such that, $p\{T(x),T(y)\} \leq \alpha p(x,y) <p(x,y)\ \forall x, y, \in X$

i.e In a contraction mapping, the distance between the images of any two points is less than the distance between the points.

Theorem: **5** .Let (X,p) be a complete metric space and let T be a contraction mapping defined on X. Then there exists one and only one point $x\in X$ such that T(x)=x, i.e, there exist a unique fixed point in X.

Proof: Let T is a contraction mapping on X there exists a real number α with $0<\alpha<1$,such that $p\{T(x),T(y)\}) \le \alpha p(x,y)$ where $x,y\in X$.

Now choose any point $x_0\in X$, Let us define a sequence $\{x_n\}$by, $x_1=T(x_o)$, $x_2=T(x_1)$, $x_3=T(x_2)$,,……………………… $x_{n1}=T(x_n)$,Then $x_n=T^n(x_0)$, $\forall\, n\in N$.

Thus the sequence $\{x_n\}$ can also be written as$\{x_0,T(x_0),T^2(x_0),\ldots\ldots\ldots T^n(x_0)\}$
Now we shall show that the sequence $\{x_n\}$ is a Cauchy sequence.

For each positive integer n, we have $P(x_n,x_{n+1})=p\{T(x_{n-1}),T(x_n)\}$

$$\begin{aligned}&\le\alpha p(x_{n-1},x_n)\\&\le\alpha p^2(x_{n-2},x_{n-1})\\&\le\alpha p^3(x_{n-3},x_{n-2})\\&\ldots\ldots\ldots\\&\ldots\ldots\ldots\\&\le\alpha p^n(x_0,x_1),\end{aligned}$$

By triangle inequality, we have $n\ge m$,

$$\begin{aligned}P(x_m,x_n) &\le p(x_m,x_{m+1}) + p(x_{m+1},x_{m+2}) + p(x_{m+2},x_{m+3}) + \ldots\ldots + p(x_{n-1},x_n)\\&\le\alpha^m p(x_0,x_1) + \alpha^{m+1}p(x_0,x_1) + \alpha^{m+2}p(x_0,x_1),+\ldots\ldots+ \alpha^{n-1}p(x_0,x_1)\\&=\alpha^m[1+\alpha+\alpha^2+\ldots\ldots+\alpha^{m-n-1}]p(x_0,x_1)\\&= \alpha^m(1-\alpha^{n-m-})p(x_0,x_1)/(1-\alpha) \quad (\text{series are in G.P, whose c.r}=\alpha)\\&< \alpha^m p(x_0,x_1)/(1-\alpha)\\&\to 0 \text{ as } m\to\infty \text{ since } \alpha<1.\end{aligned}$$

Hence $\{x_n\}$is Cauchy sequence in X.

Let $x_n\to x$, Since X is complete, so $x\in X$
Further ,since T is continuous ,therefore we have , $T(x)=T(\lim n\to\infty\ x_n)= \lim n\to\infty\ T(x_n)= \lim n\to\infty\ x_{n+1}=x$
Hence T(x)=x,This shows that x is a fixed point.

To prove uniqueness, Let T(y)=y for some $y\in X$ is another fixed point. Then T(x)=x ,T(y)=y.
Now $p(x.y)=p(Tx,Ty)\le\alpha p(x,y)$

$\le\alpha p(x_{n-1},x_n)=(1-\alpha)p(x,y)\le 0$, Since $0<\alpha<1$,hence $p(x,y)\le 0$
But p is a metric and so $p(x,y)\ge 0$
Hence p(x,y)=0 which shows that x=y
Thus there is a unique fixed point for T.

Quasi metric spaces

The mathematicians W.A. Wilson (6) J.C. Kelley(3) , Lane E.P(4) and Patty C.W. (5) have studied Quasi metric space in this study of bi-topological spaces .

Definition:6 (Quasi-metric) Let X be a non-empty set and let p:XxX$\to[0,\infty)$ be a function which satisfies

(i) $p(x, y) = 0$ iff $x = y$, $\forall\, x, y\in X$
(ii) $p(x, z)\le p(x, y) + p(y, z)$; $\forall\, x, y, z\in X$
Then p is called a Quasi-metric and the pair(X,p) is called a Quasi-metric space. For a quasi-metric p on X there exists a quasi-metric q on X, called the conjugate of p given by $q(x, y) = p(y, x)$, $\forall\, x, y\in X$.

Definition: **7 (p -Convergent)**
A sequence $\{x_n\}$ of points of X is said to be p-converge at a point $x\in X$. If $p(x, x_n)\to 0$ as $n\to\infty$ and the sequence $\{x_n\}$ is called p-convergent in X.

Definition: 8 Let (x,p) be a quasi-metric space and $\{x_n\}$ be a sequence in X,we say that$\{x_n\}$is left-cauchy if and only if for every ε>0,,there exists a positive integer N=N(ε)such that $p(x_n,x_n) < \varepsilon$ for all $m \geq n > N$.

Definition: 9 (Completeness)
A sequence $\{x_n\}$ of X is said to be p-Cauchy if and only if for ε>0, there exists a positive integer K such that $p(x_m, x_n) < \varepsilon$ form $m > n \geq K$. If every p-cauchy sequence in quasi-metric space (X, p) is p-convergent in X then (X, p) is said to be complete.

Definition:10 A self mapping T on a quasi-metric space (X, p) is said to be orbital continuous at x_0 of X if for some $x \in X_2 p(x, x_{ni}) \to 0$ implies $p(Tx, Tx_{ni}) \to 0$, where $\{x_{ni}\}$ is a subsequence of sequence $\{x_n\}$ given by $Tx_n = x_{n+1}$, n = 0, 1, 2, ...

In 1991,Chikkala R. and Baisnab A.P. [1] have proved the following fixed point theorem on complete quasi-metric spaces.

Definition: 11 Let (x,p) be a complete quasi-metric space. Let T: X → X be a self-mapping which satisfies the following conditions:

$$P(Tx, Ty) \leq \alpha\,[q(x, Tx) + q(y, Ty)] + \beta\, p(x, y) \ldots\ldots\ldots\ldots \quad (1)$$

Where $\alpha, \beta \geq 0$ such that $2\alpha + \beta < 1$

T is p-orbitally continuous at some point x_0 of x, ... (2)

Then there is a unique fixed point of T in X. We generalize it for a complete quasi-metric space

Definition: 12
Let T be a self mapping of a complete quasi-metric space (x,p) satisfying the following conditions :

$$p(Tx, Ty) \leq \alpha\,[(x, Tx) + q(y, Ty)] + \beta \frac{q(x,Tx)\,(y,Ty)}{(x,y)} + \gamma(x, y) \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Where $\alpha, \beta, \gamma \geq 0$ such that $2\alpha + \beta + \gamma < 1$

T is d-orbitally continuous at some point x_0 of X .... (4)
Then T has a unique fixed point in X.

Proof:
We define a sequence $\{x_n\}$ in X given by $x_n+1 = Tx_n$, for n = 0, 1, 2, ... Let $x_n \neq x_{n+1}$.
Then on suitable application of (1) we obtain

$$p(x_2, x_1) = p(Tx_i, Tx_0)$$
$$\leq \alpha\,[q(x_1, x_2) + q(x_0, x_1)] + \beta \frac{q(x_1,x_2)\,q(x_0,x_1)}{p(x_1,x_0)} + \gamma p(x_1, x_0) \quad = \alpha\,[p(x_2, x_1) + q(x_0, x_1)] + \beta \frac{p(x_2,x_1)\,q(x_0,x_1)}{q(x_0,x_1)} + \gamma q(x_0, x_1)$$

which implies $p(x_2, x_1) \leq hq(x_0, x_1)$,

where $h = \dfrac{\alpha+\gamma}{1-\alpha-\beta} < 1$

$$p(x_3, x_2) = p(Tx_2, Tx_1)$$
$$\leq \alpha\,[q(x_2, x_3) + q(x_1, x_2)]$$

$$+ \beta \frac{q(x_2, x_3)\ q(x_1, x_2)}{p(x_2, x_1)} + \gamma p(x_2, x_1)$$

$$= \alpha\, [p(x_3, x_2) + q\, (x_1, x_2)] + \beta \frac{p(x_3, x_2)\ q(x_1, x_2)}{q(x_1, x_2)} + \gamma q\ d(x_1, x_2)$$

Which implies $p(x_3, x_2) \le hq\, (x_1, x_2) = hp(x_2, x_1) \le h_q^2\, (x_0, x_1)$

By induction, we get

$p(x_{n+1}, x_n) \le h_q^n\, (x_0, x_1)$ ………………… (5)

Now for m > n, we get an inequality.

Now, $p(x_m, x_n) \le p(x_m, x_{m-1}) + p\, (x_{m-1}, x_{m-2}) + \ldots\ldots + p\, (x_{n+1}, x_n)$

$\le (h^{m-1} + h^{m-2} + \ldots\ldots + h^n)\, q(x_0, x_1)$

$\le \frac{h^n}{1-h} q(x_0, x_1) \to 0$ as $n \to \infty$

This shows that $\{x_n\}$ is a Cauchy sequence in complete space X and therefore there exists a point z in X such that p lim x_n = z.

By the orbital continuity of self-mapping T we get

Tz = p lim Tx_n = p lim x_{n+1} = z, which shows that z is a fixed point of T.

We now prove its uniqueness for which let ω be another fixed point of T. Then by applying the condition (1), we get

$$p(z, \omega) = p(Tz, T\omega) \le \alpha\, [\, (z, Tz) + q\, (\omega\, .\, \omega)] + \beta \frac{q(z,z)\ q(\omega,\omega)}{p(z,\omega)} + \gamma p\, (z, \omega)$$

$p(z, \omega) = p(Tz, T\omega) \le \gamma p\, (z, \omega) < (z, \omega)$

which is contradiction

Thus, T has a unique fixed point in X.

Definition: 13.

Let (X, p) be a complete quasi-metric space. Let T : X→X be a self contradiction mapping which satisfies the condition.

$$P(Tx, Ty) \le \alpha \max\{\, p(x, y), \tfrac{1}{2}\, [\, q(x, Tx) + q(y, Ty)\,], \tfrac{1}{2}\, [\, q(x, Ty) + q(y, Tx)]\, \} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

Where $\alpha \in [0, 1]$, then T has a unique fixed point.

Proof:

Let us define a sequence $\{x_n\}$ mix such that $x_n+1 = Tx_n$, for n = 0, 1, 2, … and $x_{n+1} \ne x_n$.

By suitable application of (6), we get

$p(x_2, x_1) = p(Tx_1, Tx_0)$

$\le \alpha \max\{\, p(x_1, x_0), \tfrac{1}{2}\, [q\, (x_1, x_2) + q\, (x_0, x_1)\,],$
$\tfrac{1}{2}\, [q\, (x_1, x_1) + q(x_0, x_2)]\, \}$

Hence, we obtain an inequality

$p(x_2, x_1) \le \alpha p\, (x_1, x_0) = \alpha q\, (x_0, x_1),$

$\Rightarrow \quad p(x_2, x_1) \le \frac{\alpha}{2-\alpha} p(x_0, x_1),$

or $\quad p(x_2, x_1) \le \frac{\alpha}{2} q(x_0, x_2) \le \frac{\alpha}{2} [p\, (x_2, x_1) + q\, (x_0, x_2)]$

i.e. $p(x_2, x_1) \leq \frac{\alpha}{2-\alpha} q(x_0,)$.

In each case we have $p(x_2, x_1) \leq hq(x_0, x_1)$, where $0 \leq h < 1$. The remaining part is verifiable from theorem (6).

Theorem.14:

If in theorem x_1 (5) condition (1) is replaced by

$$[p(Tx, Ty)]^2 \leq k \max \{ q(x, Tx)\, q(y, Ty), p(x, Ty)\, q(y, Tx), p(x, y)\, q(x, Tx), p(x, y)\, q(y, Ty), [p(x, y)]^2\} \quad \ldots\ldots(7)$$

where $k \in [0, 1]$, then T has a unique fixed point in X.

Proof: ***Sequence $\{x_n\}$ is defined in the same way as in theorem*** *(*6) On applying (7), we get

$$\begin{aligned}[p(x_2, x_1)]^2 &= [(Tx_1, Tx_0)]^2 \\ &\leq k \max \{ q(x_1, x_2)\, q(x_0, x_1), p(x_1, x_1)\, q(x_0, x_2), \\ &\quad p(x_1, x_0)\, q(x_1, x_2), p(x_1, x_0)\, q(x_0, x_1), \\ &\quad [p(x_1, x_0)]^2 \} \\ &= k \max \{ p(x_2, x_1)\, q(x_0, x_1), 0, q(x_0, x_1)\, p(x_2, x_1), \\ &\quad [q(x_0, x_1)]^2, [q(x_0, x_1)]^2 \}\end{aligned}$$

From the above it follows that

$[p(x_2, x_1)]^2 \leq k\, p(x_2, x_1)\, q(x_0, x_1)$ implies,

$p(x_0, x_1) \leq k^{1/2} q(x_0, x_1)$

or $[p(x_2, x_1)]^2 \leq k [q(x_0, x_1)]^2$ implies,

$p(x_2, x_1) \leq k\, q(x_0, x_1)$

As $k \in [0, 1]$ therefore $0 \leq k < k^{1/2} < 1$. Thus we have $p(x_2, x_1) \leq h\, q(x_0, x_1)$ where $h \in [0, 1]$. Rest proof is similar to that of theorem (.6). Uniqueness of the fixed point follows easily.

REFERENCES

1. Chikkaler R. and Baisnab A.P.: An analogue caristic fixed point theorem in a quasi-metric spaces. Proc. Math. Acad. Sci. India 61(A),11(1991),237-243.
2. Dhage B.C. and Dhobale V.V: Some fixed point theorem in metric spaces. Act Ciencia Inica 11(2) (1985), 138-142.
3. Kelley J.C.: Bi-topological spaces. Proc.–London Math. Soc. 13(1963),71-89.
4. Lane E.P. : Bi-topological spaces and Quasi uniform spaces. Proc. London Math. Soc. 17(1967), 241-256.
5. Patty, C.W.: Biltopological spaces Duke Math Jr. 34(1967), 387-391.
6. Wilson W.A.: On Quasi Metric spaces. Amer. Math. Jour. 36(1931), 675-684.